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AN INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEM.(U)

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AN INHOMOGENEOUS QUASILINEAR  
HYPERBOLIC SYSTEM

Ching-hua Wang

Mathematics Research Center  
University of Wisconsin-Madison  
610 Walnut Street  
Madison, Wisconsin 53706

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ABSTRACT

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We consider quasilinear hyperbolic partial differential equations modeling ideal gas flow under various physical effects. When these effects are represented as Lipschitz continuous functions of the states, solutions to the initial value problem are shown to exist globally in time. Our analysis is based on the random choice method which generalizes the Glimm scheme for hyperbolic conservation laws.

When the effects are strongly dissipative then the flow decays exponentially to a constant state as time tends to infinity.

AMS (MOS) Subject Classifications: 35L60, 35L67, 35B40

Key Words: Quasilinear, Initial value problem, Riemann Invariants, Weak solution, Existence of theorem of global solution, Asymptotic behavior, Inhomogeneous.

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## SIGNIFICANCE AND EXPLANATION

We study the initial value problem for quasilinear hyperbolic system of the following form:

$$\frac{\partial u}{\partial t} + \frac{\partial (\frac{1}{v})}{\partial x} = U(u, v, x, t)$$
$$\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = V(u, v, x, t) .$$

The system models ideal gas flow where  $U$  and  $V$  represent physical effects such as damping and external moving force. Suppose that the initial data  $(u(x,0), v(x,0))$  have bounded total variation, and  $U$  and  $V$  are Lipschitz continuous functions of  $u$  and  $v$ . Then the initial value problem has an admissible weak solution. When  $U$  and  $V$  represent strong dissipative effects, then the solution decays to a constant state as  $t$  tends to infinity. Our analysis is based on the random choice method which generalizes the Glimm scheme for conservation laws.

Our assumptions are general enough to include several important physical effects. Moreover, the random choice method can be applied to more general quasilinear hyperbolic systems as an effective numerical scheme.

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The responsibility for the wording and views expressed in this descriptive summary lies with M.C., and not with the author of this report.

# AN INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEM

Ching-hua Wang

## INTRODUCTION

We consider the initial value problem for the inhomogeneous hyperbolic system

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial (\frac{1}{v})}{\partial x} = U(u, v, x, t) \\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = V(u, v, x, t) \end{cases} \quad (0.1)$$

$$(u, v)(x, 0) = (u_0(x), v_0(x)) \quad (0.2)$$

Here  $u_0(x)$  and  $v_0(x)$  are bounded functions with bounded total variation,  $v_0(x)$  has a positive lower bound. It is assumed that there exists a constant  $M > 0$  such that  $U$  and  $V$  do not depend on  $x$  for  $|x| > M$ .  $U$  and  $V$  are smooth functions of  $u, v, x$  and  $t$ .

By the Riemann invariants

$$r = u + \log v \quad s = u - \log v \quad (0.3)$$

the system (0.1) can be diagonalized as

$$\begin{cases} \frac{\partial r}{\partial t} - \frac{1}{v} \frac{\partial r}{\partial x} = R(r, s, x, t) \\ \frac{\partial s}{\partial t} + \frac{1}{v} \frac{\partial s}{\partial x} = S(r, s, x, t) \end{cases} \quad (0.4)$$

where  $R(r, s, x, t) = U + \frac{V}{v}$ ,  $S(r, s, x, t) = U - \frac{V}{v}$ ;  $(U, V) = (U, V)(\frac{r+s}{2}, \exp(\frac{r-s}{2}), x, t)$

Throughout this paper we also assume that there exist two constants  $K_1, K_2$  such that

$$|R(r_1, s_1, x, t) - R(r_2, s_2, x, t)| \leq K_1(|r_1 - r_2| + |s_1 - s_2|) \quad (0.5)$$

$$|S(r_1, s_1, x, t) - S(r_2, s_2, x, t)| \leq K_1(|r_1 - r_2| + |s_1 - s_2|)$$

$$|\frac{\partial R}{\partial x}| \leq K_2, \quad |\frac{\partial S}{\partial x}| \leq K_2 \quad (0.6)$$

When  $(U, V) \equiv (0, 0)$ , system (0.1) reduces to the following system

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial(\frac{1}{v})}{\partial x} = 0 \\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0 \end{cases} \quad (0.7)$$

The initial boundary value problem of (0.7) has been solved by Nishida T. [6]. When  $U$  and  $V$  do not depend on  $x$  and  $t$ , the global solution of (0.1 and (0.2) has been constructed by Ying Lung-an and Wang Ching-hua [7]. For general inhomogeneous system

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(x, t, u) \quad (0.8)$$

Ying Lung-an and Wang Ching-hua [8] have proved the existence of the global solution of initial value problem of system (0.8) for  $g(x, t, u) = g(u) e^{-Kt}$ . When  $g$  does not depend on  $t$ , Liu Tai-Ping [5] has constructed global solutions for system (0.8) and studied their asymptotic behavior.

In general (0.1) and (0.2) does not possess smooth solutions, and we look for weak solutions in  $0 \leq t < T$ , i.e., solutions satisfying

$$\iint_{0 \leq t < T} (u \phi_t + \frac{1}{v} \phi_x + U \phi) dx dt + \int_{-\infty}^{+\infty} u_0(x) \phi(x, 0) dx = 0 \quad (0.9)$$

$$\iint_{0 \leq t < T} (v \psi_t - u \psi_x + V \psi) dx dt + \int_{-\infty}^{+\infty} v_0(x) \psi(x, 0) dx = 0 \quad (0.10)$$

for any smooth functions  $\phi$  and  $\psi$  with compact support in  $0 \leq t < T$ .

The purpose of this paper is to prove the existence of the weak solution for (0.1) and (0.2) and study their asymptotic behavior as the time variable  $t$  tends to infinity.

# 1. Difference Scheme

We now describe the difference scheme we use for solving (0.1) and (0.2), which is the generalization of one in [2] and [7]. Randomly choose an equidistributed sequence  $a=\{a_i\}$  in  $(-1,1)$  and mesh length  $\Delta x=l$   $\Delta t=h$  satisfying the Courant-Friedrich-Lewy condition

$$\frac{l}{h} = \text{const} \geq \frac{1}{v}$$

for all  $v$  under consideration.

Let  $(\tilde{u}, \tilde{v})(x, t, m+1, k, u_l, v_l, u_r, v_r)$  denote the solution of the system (0.7) at the point  $((m+1)l, kh)$  with the following initial data

$$(u, v)(x, kh) = \begin{cases} (u_l, v_l) & x < (m+1)l \\ (u_r, v_r) & x > (m+1)l \end{cases} \quad (1.1)$$

We take

$$(u, v)(x, t, m+1, k, u_l, v_l, u_r, v_r) = (\tilde{u} + U(\tilde{u}, \tilde{v}, (m+1)l, kh)(t-kh),$$

$$\tilde{v} \exp \left( \frac{V(\tilde{u}, \tilde{v}, (m+1)l, kh)}{\tilde{v}} (t-kh) \right) \quad (1.2)$$

as the approximation solution of the system (0.1) with the initial data (1.1). In fact, using the Riemann invariants, we can rewrite (1.2) in the following form

$$(r, s) = (\tilde{r} + R(\tilde{r}, \tilde{s}, (m+1)l, kh)(t-kh), \tilde{s} + S(\tilde{r}, \tilde{s}, (m+1)l, kh)(t-kh)) \quad (1.3)$$



The construction of the difference solution  $(u,v)(x,t)$  proceeds as follows:

For each mesh length  $\ell$ , let  $(u_{\ell,0}(x), v_{\ell,0}(x))$  be defined by the equation

$$(u_{\ell,0}, v_{\ell,0})(x) = \begin{cases} (u_0(-\infty), v_0(-\infty)), & x < -(2[\frac{1}{\ell}]^2 + 1)\ell \\ (u_0(m\ell), v_0(m\ell)), & \text{for } |x| \leq (2[\frac{1}{\ell}]^2 + 1)\ell \text{ and} \\ & (m-1)\ell < x < (m+1)\ell, m=\text{even} \\ (u_0(+\infty), v_0(+\infty)), & x > ((2[\frac{1}{\ell}]^2 + 1)\ell \end{cases} \quad (1.4)$$

Then we set

$$(u,v)(x,t) = (u,v)(x,t, m+1,0, (u_{\ell,0}, v_{\ell,0})(m\ell), (u_{\ell,0}, v_{\ell,0})(m+2)\ell)$$

$$\text{for } m\ell < x < (m+2)\ell, 0 < t < h, m = \text{even} \quad (1.5)$$

Inductively suppose that  $(u,v)(x,t)$  has been defined for  $0 \leq t < kh$ , we set

$$(u,v)(x,t) = (u,v)(x,t, m+1,k, (u,v)((m+a_k)\ell, kh-0), (u,v)((m+2+a_k)\ell, kh-0))$$

$$\text{for } m\ell < x < (m+2)\ell, kh \leq t < (k+1)h, m+k = \text{even} \quad (1.6)$$

It is obvious that  $(u,v)(x,t)$  depends on the mesh length  $h$  and sequence  $a = \{a_i\}$ , but we omit the subscript.

## 2. Bounds for the Difference Equation

Because we use  $(u, v)$  defined in (1.2) as the building block to build the difference solution of (0.1) and (0.2), we have to do some estimates on it.

First we consider  $(\tilde{u}, \tilde{v}) (x, t, (m+1), k, u_\ell, v_\ell, u_r, v_r)$  defined in Section 1. It is well known that there are three constant region in the half plane  $t > kh$ , in which  $(\tilde{u}, \tilde{v})$  equals to  $(u_\ell, v_\ell)$ ,  $(u_m, v_m)$  and  $(u_r, v_r)$  respectively.  $(u_m, v_m)$  is connected to  $(u_\ell, v_\ell)$  by a 1-wave on the left and  $(u_r, v_r)$  is connected to  $(u_m, v_m)$  by a 2-wave on the left. We define  $(r_\ell, s_\ell)$ ,  $(r_m, s_m)$  and  $(r_r, s_r)$  by means of (0.3) and define

$$\Delta r = r_m - r_\ell \quad \Delta s = s_r - s_m \quad (2.1)$$

$$P((u_\ell, v_\ell), (u_r, v_r)) = -\min(0, \Delta r) - \min(0, \Delta s) \quad (2.2)$$

It has been proved [6] that

$$P((u_i, v_\ell), (u_j, v_j)) < P((u_i, v_i), (u_k, v_k)) + P((u_k, v_k), (u_j, v_j)) \quad (2.3)$$

where  $u_i, u_k, u_j$  are any constants,  $v_i, v_k, v_j$  are any positive constants

Let

$$\delta r = r_r - r_\ell, \quad \delta s = s_r - s_\ell \quad (2.4)$$

we have

$$\begin{aligned}\delta r_t &= u_r + U_r(t-kh) + \log(v_r \exp \frac{v_r}{v_r}(t-kh)) - (u_l + U_l(t-kh) + \log(v_l \exp \frac{v_l}{v_l}(t-kh))) \\ &= (r_r - r_l) + (R_r - R_l)(t-kh)\end{aligned}$$

$$\begin{aligned}\delta s_t &= u_r + U_r(t-kh) - \log(v_r \exp \frac{v_r}{v_r}(t-kh)) - (u_l + U_l(t-kh) - \log(v_l \exp \frac{v_l}{v_l}(t-kh))) \\ &= (s_r - s_l) + (S_r - S_l)(t-kh)\end{aligned}$$

where  $U_r, V_r, R_r, S_r$  evaluated at  $(u_r, v_r, (m+1)l, kh)$  and  $U_l, V_l, R_l, S_l$  evaluated at  $(u_l, v_l, (m+1)l, kh)$ .

Therefore by (0.5) we have

$$\begin{aligned}\text{Lemma 2.1} \quad & \begin{cases} |\delta r_t - \delta r| \leq K_1(|\delta r| + |\delta s|)(t-nh) \\ |\delta s_t - \delta s| \leq K_1(|\delta r| + |\delta s|)(t-nh) \end{cases} \quad (2.5)\end{aligned}$$

In the same way as Lemma 3 in [7], we have

Lemma 2.2 If  $(u_r, v_r)$  can be connected to  $(u_l, v_l)$  by only one wave,  $u_r^*, u_l^*$  are any constants and  $v_r^*, v_l^*$  are any positive constants then

$$P((u_l^*, v_l^*), (u_r^*, v_r^*)) \leq P((u_l, v_l), (u_r, v_r)) + 3(|\delta r^* - \delta r| + |\delta s^* - \delta s|) \quad (2.6)$$

where  $\delta r^* = r_r^* - r_l^* = (u_r^* + \log v_r^*) - (u_l^* + \log v_l^*)$ ,  $\delta s^* = (u_r^* - \log v_r^*) - (u_l^* - \log v_l^*)$ .

In order to estimate the variations of  $r(x, t)$  and  $s(x, t)$ , we define some functionals.

For integers  $k \geq 1$ , we define

$$F(kh+0) = \sum_{m+k=\text{even}} P((u, v)((m+a_k)l, kh-0), (u, v)((m+2+a_k)l, kh-0)),$$

and for integer  $k \geq 0$

$$F((k+1)h-0) = \sum_{m+k=\text{even}} \{P((u,v)((m-1+a_{k+1})l, (k+1)h-0), \\ (u,v)(ml-0, (k+1)h-0)) +$$

$$P((u,v)(ml-0, (k+1)h-0), (u,v)(ml+0, (k+1)h-0)) +$$

$$P((u,v)(ml+0, (k+1)h-0), (u,v)((m+1+a_{k+1})l, (k+1)h-0))\}$$

Lemma 2.3. For any given  $T > 0$ ,  $\sum_{-\infty}^{+\infty} u(\cdot, t)$  and  $\sum_{-\infty}^{+\infty} v(\cdot, t)$  the total variation of the difference solution  $(u(x, t), v(x, t))$ , are bounded uniformly for the mesh length  $h$ , sequence  $a$  and  $t < T$ ,  $v(x, t)$  has positive lower bound uniformly for the same parameters provided  $k < \frac{1}{2K_1}$ .

Proof. Since  $U(u, v, x, t)$ ,  $V(u, v, x, t)$  do not depend on  $x$  for  $|x| > M$  and (0.5) there exists a constant  $\bar{M}$  depending only on  $T$  and  $u_0(\pm\infty)$ ,  $v_0(\pm\infty)$  such that  $|s(\pm\infty, t)| \leq \bar{M}$ ,  $|r(\pm\infty, t)| \leq \bar{M}$  for  $0 \leq t < T$ , provided  $h$  is small enough.

It follows from that  $U(u, v, x, t)$ ,  $V(u, v, x, t)$  do not depend on  $x$  for  $|x| > M$  again and (0.6) that

$$\sum_{m+k=\text{even}} P((u,v)(ml-0, (k+1)h-0), (u,v)(ml+0, (k+1)h-0)) \leq$$

$$\leq h \sum_{m+k=\text{even}} \{|R((r,s)((m+a_k)l, kh-0), (m-1)l, kh) - R((r,s)(m+a_k)l, kh-0),$$

$$(m+1)l, kh) + |S((r,s)((m+a_k)l, k-0), (m-1)l, kh) - S((r,s)(m+a_k)l, kh-0),$$

$$(m+1)l, kh)|\} \leq 4K_2 Mh.$$

It is obvious that  $\int_{-\infty}^{+\infty} r(\cdot, nh+0) + \int_{-\infty}^{+\infty} s(\cdot, nh+0) \leq 4F(nh+0) + 4\bar{M}$ .  
Then from Lemma 2.1, Lemma 2.2, we obtain

$$\begin{aligned} F((n+1)h-0) &\leq F(nh+0) + 6K_1 h \left( \int_{-\infty}^{+\infty} r(\cdot, nh+0) + \int_{-\infty}^{+\infty} s(\cdot, nh+0) \right) + 4K_2 Mh \\ &\leq F(nh+0) (1+24K_1 h) + (24K_1 \bar{M}h + 4K_2 Mh) \end{aligned}$$

Because  $F((n+1)h+0) \leq F((n+1)h-0)$  we have

$$F((n+1)h+0) \leq F(nh+0) (1+24K_1 h) + 24K_1 \bar{M}h + 4K_2 Mh$$

Let  $F(0) = \sum_{m=\text{even}} P((u_0, v_0)(m\ell), (u_0, v_0)((m+2)\ell))$ , then we obtain

$$F(0) \leq \int_{-\infty}^{+\infty} r_0(\cdot) + \int_{-\infty}^{+\infty} s_0(\cdot), \text{ where } r_0(x) = u_0(x) + \log v_0(x),$$

$$s_0(x) = u_0(x) - \log v_0(x).$$

After simple computation we get

$$F((n+1)h+0) \leq (F(0) + \bar{M} + \frac{K_2 M}{6K_1}) e^{24K_1 T}$$

$$\leq \left( \int_{-\infty}^{+\infty} r_0(\cdot) + \int_{-\infty}^{+\infty} s_0(\cdot) + \bar{M} + \frac{K_2 M}{6K_1} \right) e^{24K_1 T} \text{ for } (n+1)h < T$$

and thus

$$\begin{aligned} \int_{-\infty}^{+\infty} r(\cdot, nh+0) + \int_{-\infty}^{+\infty} s(\cdot, nh+0) &\leq 4 \left( \int_{-\infty}^{+\infty} r_0(\cdot) + \int_{-\infty}^{+\infty} s_0(\cdot) + \bar{M} + \frac{K_2 M}{6K_1} \right) e^{24K_1 T} + \\ &\quad + 4\bar{M} \text{ for } nh < T \end{aligned}$$

It follows from that  $u_0(x)$  and  $r_0(x)$  are bounded functions with bounded variation and  $v_0(x)$  has a positive bound, Lemma 2.1 and  $h < \frac{1}{2K_1}$  that

$$\int_{-\infty}^{+\infty} r(\cdot, t) + \int_{-\infty}^{+\infty} s(\cdot, t) \leq Q \quad \text{for } t < T \quad (2.7)$$

$$\text{where } Q = 8 \left( \int_{-\infty}^{+\infty} r_0(\cdot) + \int_{-\infty}^{+\infty} s_0(\cdot) + \bar{M} + \frac{K_2 M}{6K_1} \right) e^{24K_1 T} + \frac{K_2}{K_1} M + 8\bar{M}.$$

Therefore

$$|r(x, t)| \leq 2\bar{M} + Q, \quad |s(x, t)| \leq 2\bar{M} + Q, \quad |u(x, t)| \leq 2\bar{M} + \frac{Q}{2} \quad \text{for } t < T \quad (2.8)$$

$$e^{-(2\bar{M} + \frac{Q}{2})} \leq v(x, t) \leq e^{2\bar{M} + \frac{Q}{2}} \quad \text{for } t < T \quad (2.9)$$

$$\int_{-\infty}^{+\infty} u(\cdot, t) \leq Q/2 \quad \text{for } t < T \quad (2.10)$$

$$\int_{-\infty}^{+\infty} v(\cdot, t) \leq \frac{1}{2} Q e^{2\bar{M} + \frac{Q}{2}} \quad \text{for } t < T \quad (2.11)$$

Q.E.D.

Lemma 2.4. For any given  $X > 0$ , there exists constant  $D$  depending only on  $U, V, u_0(x), v_0(x), T$  and  $X$  such that

$$I(X) = \int_{-X}^X (|u(x, t_2) - u(x, t_1)| + |v(x, t_2) - v(x, t_1)|) dx \leq \\ \leq D((t_2 - t_1) + h), \quad 0 < t_1, t_2 < T \quad (2.12)$$

provided  $h$  is small enough.

Proof: For definiteness suppose  $t_1 < t_2$  and there is a pair of integers  $n, k$  such that  $nh \leq t_1 < (n+1)h < \dots < (n+k)h < t_2 \leq (n+k+1)h$ . It is obvious that

$$I(X) \leq I_1(X) + I_2(X)$$

where

$$\begin{aligned} I_1(X) &= \sum_{i=0}^{k+1} \int_{-X}^X (|u(x, (n+i)h+0) - u(x, (n+i)h-0)| + \\ &\quad + |v(x, (n+i)h+0) - v(x, (n+i)h-0)|) dx, \\ I_2(X) &= \sum_{i=0}^k \int_{-X}^X (|u(x, (n+i+1)h-0) - u(x, (n+i)h+0)| + \\ &\quad + |v(x, (n+i+1)h-0) - v(x, (n+i)h+0)|) dx \\ &\quad + \int_{-X}^X (|u(x, t_1) - u(x, nh+0)| + |v(x, t_1) - v(x, nh+0)|) dx + \\ &\quad + \int_{-X}^X (|u(x, t_2) - u(x, (n+k)h+0)| + |v(x, t_2) - v(x, (n+k)h+0)|) dx. \end{aligned}$$

It is not difficult to show that  $I_1(X) \leq 2\ell \left( \left\lceil \frac{t_2 - t_1}{h} \right\rceil + 3 \right)$

$$\left\{ \sup_{0 < t < T} \sup_{-\infty}^{+\infty} u(\cdot, t) + \sup_{0 < t < T} \sup_{-\infty}^{+\infty} v(\cdot, t) \right\}$$

For  $x \in ((m-1)\ell, (m+1)\ell)$ ,  $m+n+i+1 = \text{even}$ . Using (1.2) and (2.9) we have

$$|u(x, (n+i+1)h-0) - u(x, (n+i)h+0)| \leq \frac{(m+1)l}{(m-1)l} V u(x, (n+i)h+0) + |U^*| h.$$

$$|y(x, (n+i+1)h-0) - v(x, (n+i)h+0)| \leq \frac{(m+1)l}{(m-1)l} V r(x, (n+i)h+0) + 2|V^*| h$$

provided  $h$  is small enough, where

$$(U^*, V^*) = ((U, V), (x, (n+i+1)h-0), ml, (n+i)h).$$

For  $\int_{-X}^X (|u(x, t_1) - u(x, nh+0)| + |v(x, t) - v(x, nh+0)|) dx$  and  $\int_{-X}^X (|u(x, t_2) - u(x, (n+k)h+0)| + |v(x, t_2) - v(x, (n+k)h+0)|) dx$  we have similar estimate.

Combining the above estimates, we obtain

$$I_2 \leq \left( 2 \sup_{0 < t < T} \sup_{-\infty}^{+\infty} V u(\cdot, t) + \sup_{0 < t < T} \sup_{-\infty}^{+\infty} V r(\cdot, t) \right) +$$

$$+ 2 X h \left( \sup_{m+n+i+1=\text{even}} U^* + 2 \sup_{m+n+i+1=\text{even}} V^* \right) \left[ \left[ \frac{t_2 - t_1}{h} \right] + 4 \right]$$

Since (2.8) (2.9) (2.10) and (2.11), we obtain (2.12). Q.E.D.



### 3. Convergence of the Difference Solutions

Because  $(u, r)(x, t)$  is not an exact solution of (0.1) in the strip  $kh \leq t < (k+1)h, k = 0, 1, \dots, \frac{T}{h}$ , so in order to prove the existence of the weak solution of (0.1) and (0.2) we first prove two lemmas.

**Lemma 3.1.** If  $x' = \xi_1 t'$  is a 1-shock wave of  $(\tilde{u}, \tilde{v})$   $(x, t, (m+1)h, k, (u_\ell, v_\ell), (u_r, v_r))$  in the domain  $\{(x, t) : m\ell < x < (m+2)\ell, kh \leq t < (k+1)h, m+k = \text{even}\}$  then

$$|\xi_1[u] - [\frac{1}{v}]| \leq wt'(|u_m - u_\ell| + |v_m - v_\ell|) \quad (3.1)$$

$$|\xi_1[v] - [u]| \leq wt'(|u_m - u_\ell| + |v_m - v_\ell|) \quad (3.2)$$

provided  $h$  is small enough, where  $x' = x - (m+1)\ell, t' = t - kh$ , the constant  $w$  depends only on  $U, V, u_0(x), r_0(x); (u_\ell, v_\ell) = (u, v)((m+a_k)\ell, kh-0), (u_r, v_r) = (u, v)((m+2+a_k)\ell, kh-0); (u_m, v_m)$  is the constant solution being connected to  $(u_\ell, v_\ell)$  by 1-shock wave on the left;  $[ ]$  denotes the jump of the quantity in bracket across the 1-shock wave.

For  $x' = \xi_2 t'$  is a 2-shock wave of  $(\tilde{u}, \tilde{v})$ , we have same kind of result.

**Proof:** It follows from that  $(u_r, v_r)$  and  $(u_m, v_m)$  satisfy Rankine-Hugoniot condition and (0.5) (2.8) (2.11) that

$$|\xi_1[u] - [\frac{1}{v}]| \leq \left| \xi_1 (u_m - u) t' - \left( \left( \frac{1}{v_m e^{\frac{v_m t'}{v_\ell}}} - \frac{1}{v_m} \right) - \left( \frac{1}{v_\ell e^{\frac{v_\ell t'}{v_\ell}}} - \frac{1}{v_\ell} \right) \right) \right| \leq$$

$$\begin{aligned}
&\leq |\xi_1 (U_m - U_\ell) t'| + \left| \frac{1}{v_\ell} \int_0^{\frac{v_\ell t'}{v_\ell}} e^{-\eta} d\eta - \frac{1}{v_m} \int_0^{\frac{v_m t'}{v_m}} e^{-\eta} d\eta \right| \leq \\
&\leq |\xi_1 (U_m - U_\ell) t'| + |(v_\ell - v_m) \left( -\frac{1}{v_\ell^2} \int_0^{\frac{v_\ell t'}{v_\ell}} e^{-\eta} d\eta + \frac{t'}{v_\ell} e^{-\frac{v_\ell t'}{v_\ell}} \frac{\partial}{\partial v} \left( \frac{v}{v_\ell} \right) \right)| + \\
&+ \left| \frac{1}{v_m} \int_0^{\frac{\tilde{v} t'}{v_m}} e^{-\eta} d\eta \right| \leq w t' (|u_m - u_\ell| + |v_m - v_\ell|)
\end{aligned}$$

where  $(U_\ell, v_\ell) = (U, V)(u_\ell, v_\ell, (m+1)\ell, kh)$ ,  $(U_m, v_m) = (U, V)(u_m, v_m, (m+1)\ell, kh)$

$v_\star = V(u_\ell, v_\star, (m+1)\ell, kh)$ ,  $\tilde{v} = V(u_\ell, v_m, (m+1)\ell, kh)$ ;

$v_\star = \theta u_m + (1-\theta)v_\ell$ ,  $0 < \theta < 1$ . The proof of (3.2) is similar. Q.E.D.

**Lemma 3.2.** If  $(\tilde{u}, \tilde{v})$  is a 1-rarefaction wave in the subdomain

$\Omega_1 = \{(x, t); \xi_0' < \frac{x'}{t'} < \xi_1', 0 < t' < h\}$  of the domain  $\Omega = \{(x, t)$ ,

$m\ell < x < (m+2)\ell, kh \leq t < (k+1)h, m+k=\text{even}\}$ ; then

$$\begin{aligned}
&\left| \int_{\Omega_1} (u \phi_t + \frac{1}{v} \phi_x + U) dx dt + \int_{\partial \Omega_1} u \phi ds - \frac{1}{v} \phi dt \right| \\
&\leq R_1 h^3 + R_2 h^2 \left( \frac{\xi_1' t'}{\xi_0' t'} u(kh+0) + \frac{\xi_1' t'}{\xi_0' t'} v(\cdot, kh+0) \right) \quad (3.3)
\end{aligned}$$

$$\begin{aligned}
&\left| \int_{\Omega_1} (v \chi_t - u \chi_x + V) dx dt + \int_{\partial \Omega_1} v \chi dx - u \chi dt \right| \\
&\leq R_1 h^3 + R_2 h^2 \left( \frac{\xi_1' t'}{\xi_0' t'} u(\cdot, kh+0) + \frac{\xi_1' t'}{\xi_0' t'} v(\cdot, kh+0) \right) \quad (3.4)
\end{aligned}$$

provided  $h$  is small enough.

Where  $x' = x - (m+1)l$ ,  $t = t - kh$ ;  $(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v})(x, t, (m+1)l, k, (u, v)(m+1)l, kh-0), (u, v)((m+2+a_k)l, kh-0))$ ,  $\phi$  and  $\chi$  are smooth functions with compact support in  $0 \leq t < T$ ;  $R_1$  and  $R_2$  are constants depending only on  $U, V, u_0(x), r_0(x), \phi$  and  $\chi$ ,  $(U, V) = (U(u, v, x, t), V(u, v, x, t))$ .

For  $(\tilde{u}, \tilde{v})$  is a 2-rarefaction wave in the subdomain  $\Omega_2 = \{(x, t); \xi'_1 < \frac{x'}{t} < \xi_2, 0 < t < h\}$  of the domain  $\Omega$ , we have same kind of result.

Proof:  $(\tilde{u}, \tilde{v})$ , as a 1-rarefaction wave in  $\Omega_1$ , satisfies (0.7) in  $\Omega_1$  in classic sense. Because of (0.5) and (2.9),  $U$  and  $V$ , as the functions of  $u$  and  $v$ , satisfy Lifshitz condition with the Lifshitz constant depending only on  $K_1, \bar{M}$  and  $Q$ .

It follows from (1.2)(1.6) that

$$\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{U}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{U}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial x} ) t' \quad (3.5)$$

$$\frac{\partial u}{\partial t} = \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{U}}{\partial \tilde{u}} \frac{\partial u}{\partial t} + \frac{\partial \tilde{U}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial t} ) t' + \tilde{U} \quad (3.6)$$

$$\begin{aligned} \frac{\partial v}{\partial x} = \frac{\partial \tilde{v}}{\partial x} e^{\frac{\tilde{V} t'}{\tilde{v}}} + (\tilde{v})^{-1} t' e^{\frac{\tilde{V} t'}{\tilde{v}}} (\tilde{v} \frac{\partial \tilde{v}}{\partial u} \frac{\partial \tilde{u}}{\partial x} + \\ + \tilde{v} \frac{\partial \tilde{V}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial x} - \tilde{V} \frac{\partial \tilde{v}}{\partial x}) \quad (3.7) \end{aligned}$$

$$\frac{\partial v}{\partial t} = \frac{\partial \tilde{v}}{\partial t} e^{\frac{(\tilde{v}t')}{\tilde{v}}} + (\tilde{v})^{-1} t' e^{\frac{(\tilde{v}t')}{\tilde{v}}} \left( \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial t} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial t} - \tilde{v} \frac{\partial \tilde{v}}{\partial t} \right) + \tilde{v} e^{\frac{(\tilde{v}t')}{\tilde{v}}} \quad (3.8)$$

where  $(\tilde{U}, \tilde{V}) = (\tilde{U}, \tilde{V})(\tilde{u}, \tilde{v}, (m+1)l, kh)$ .

Let  $J_1(x, t) = \frac{\partial u}{\partial t} + \frac{\partial(\frac{1}{v})}{\partial x} - U$ , then we obtain

$$\begin{aligned} J_1(x, t) &= \frac{\partial \tilde{u}}{\partial t} + \left( \frac{\partial \tilde{u}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{u}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial t} \right) t' + \tilde{U} - (\tilde{v})^{-2} e^{-\frac{(\tilde{v}t')}{\tilde{v}}} \\ &\quad \left( \frac{\partial \tilde{v}}{\partial x} e^{\frac{(\tilde{v}t')}{\tilde{v}}} + (\tilde{v})^{-1} t' e^{\frac{(\tilde{v}t')}{\tilde{v}}} \left( \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial x} - \tilde{v} \frac{\partial \tilde{v}}{\partial x} \right) - \right. \\ &\quad \left. U = (1 - e^{\frac{(\tilde{v}t')}{\tilde{v}}}) (\tilde{v})^{-2} \frac{\partial \tilde{v}}{\partial x} + t' \left( \frac{\partial \tilde{v}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{u}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial t} \right) - t' (\tilde{v})^{-3} e^{\frac{(\tilde{v}t')}{\tilde{v}}} \right. \\ &\quad \left. \left( \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial x} - \tilde{v} \frac{\partial \tilde{v}}{\partial x} \right) \right). \end{aligned}$$

Using (0.5) and (0.6) for estimating  $\tilde{U} - U$ , then we get

$$\begin{aligned} &\iint_{\Omega_1} (u \phi_t + \frac{1}{v} \phi_x + U \phi) dx dt + \int_{\partial \Omega_1} u \phi dx - \frac{1}{v} \phi dt = \iint_{\Omega} J_1(x, t) \phi(x, t) dx dt \leq \\ &\leq R_1 h^3 + R_2 h^2 \left( \frac{\xi_1 t'}{v} u(\cdot, kh+0) + \frac{\xi_1 t'}{v} v(\cdot, kh+0) \right). \end{aligned}$$

The proof of (3.4) is similar.

Q.E.D.

We use the difference scheme introduced in section 2 and take the ratio of the mesh lengths  $\frac{l_i}{h_i} = e^{2\bar{M}+Q}$  and  $h_i = \frac{T}{2^i}$ . When  $i$  is large enough, the difference solution  $(u, v)(x, t)$  can be defined in  $0 \leq t < T$ .

The difference solution  $(u, v(x, t))$  is discontinuous at the segment  $\{(x, t): x = m\ell_i, kh_i < t \leq (k+1)h_i, m+k=\text{even}\}$ . But it follows from (0.6), (2.8) and (2.11) that

$$\left| \int_{kh_i}^{(k+1)h_i} (u(m\ell_i+0, t) - u(m\ell_i-0, t)) \chi dt \right| \leq \int_{kh_i}^{(k+1)h_i} (t - kh_i) |U((u, v)((m+a_k)\ell_i, (m+1)\ell_i, kh_i) - U((u, v)((m+a_k)\ell_i, (m-1)\ell_i, kh_i)) \chi dt| \leq E h_i^3 \quad (3.9)$$

$$\left| \int_{kh_i}^{(k+1)h_i} \left( \frac{1}{v(m\ell_i+0, t)} - \frac{1}{v(m\ell_i-0, t)} \right) \phi dt \right| \leq E h_i^3 \quad (3.10)$$

provided  $i$  is large enough, where  $E$  is a constant depending only on  $K_2$ ,  $\bar{M}$  and  $Q$ .

With Lemma 3.1, lemma 3.2 and (3.9), (3.10) in mind by the way similar to these in the section 4 in [7] we obtain the following existence theorem.

**Theorem 3.1.** Suppose that  $u_0(x)$  are bounded functions with bounded variation,  $v_0(x)$  has a positive lower bound.  $U(u, v, x, t)$ ,  $V(u, v, x, t)$  are smooth functions satisfying (0.5) (0.6) and do not depend on  $x$  as  $|x| > M$ , where  $M$  is any given positive number, then for any given  $T > 0$  the initial value problem (0.1) and (0.2) has a weak solution in  $0 < t < T$ .

#### 4. Asymptotic behavior

Now we consider the asymptotic behavior of the solution of (0.1) and (0.2) under the following additional restrictions on  $U$  and  $V$ :

$$R \text{ depends only on } r, \quad \text{and} \quad R'(r) \leq -\alpha; \quad (4.1)$$

$$S \text{ depends only on } s, \quad \text{and} \quad S'(s) \leq -\alpha \quad (4.2)$$

where  $\alpha$  is a positive number.

Lemma 4.1. Suppose that  $R(r)$  satisfies (4.1), then the initial value problem

$$\begin{cases} \frac{dr}{dt} = R(r) \\ r(0) = r_0 \end{cases} \quad (4.3)$$

$$r(0) = r_0 \quad (4.4)$$

possesses a unique solution  $r(t, r_0)$  satisfying

$$|r(t, r_0) - \bar{r}| \leq |r_0 - \bar{r}| e^{-\alpha t} \quad \text{for } t > 0 \quad (4.5)$$

where  $\bar{r}$  is the unique solution of  $R(r) = 0$ ; and for

$$r_h(t, r_0) = \begin{cases} r_0 + R(r_0)t & , \quad 0 \leq t \leq h \\ r_h(h, r_0) + R(r_h(h, r_0))(t-h) & , \quad h \leq t \leq 2h \\ \dots & \\ r_h(nh, r_0) + R(r_h(nh, r_0))(t-nh) & , \quad nh \leq t \leq (n+1)h \\ \dots & \end{cases} \quad , \quad \text{the Euler's}$$

difference solutions of (4.3) (4.4), the following inequality holds

$$|r_h(t, r_0) - \bar{r}| \leq |\bar{r} - r_0| e^{-\alpha t} \quad \text{for } t > 0 \quad (4.6)$$

provided  $h < \frac{1}{\alpha}$ .

Proof. Since the assumptions (0.5), (4.3) and (4.4) possesses a unique solution  $r(t, r_0)$  for all  $t > 0$ . By (4.1) we have

$$\frac{d(r(t, r_0) - \bar{r})}{dt} = R(r(t, r_0)) = -\alpha(r(t, r_0) - \bar{r})$$

and therefore (4.5).

Now we turn to prove (4.6).

Without loss of generality we assume  $r_0 \neq \bar{r}$ , because of  $r_h(t, \bar{r}) \equiv r(t, \bar{r})$ .

For definiteness suppose  $r_0 > \bar{r}$ . It follows from (0.5) (4.1) that for any  $\tilde{r} > \bar{r}$  and  $t_0 > 0$  the initial value problem

$$\begin{cases} \frac{dr}{dt} = R(r) \\ r|_{t=t_0} = \tilde{r} \end{cases} \quad \text{has a unique solution } r(t, \tilde{r}, t_0) \text{ such that}$$

$$\frac{d^2 r(t, \tilde{r}, t_0)}{dt^2} > 0; \text{ therefore}$$

$$\tilde{r} < r_h(t, r_0) < r(t, r_0) \quad \text{for } t > 0 \quad (4.7)$$

provided  $h < \frac{1}{\alpha}$ .

Similary for  $r_0 < \bar{r}$  we have

$$r(t, r_0) < r_h(t, r_0) < \bar{r} \quad \text{for } t > 0 \quad (4.8)$$

By (4.5) (4.7) (4.8) we obtain (4.6).

Lemma 4.2. Under the assumption (4.1) and (4.2) we have

$$F(k+1)h-0) \leq (1-(1-c)\alpha h)F(kh+0), \text{ for } k \geq 0 \quad (4.9)$$

provided  $h < \min(\frac{1}{\alpha} \frac{1}{2K_1})$ , where  $c$  is a constant depending only on  $\lim_{x \rightarrow -\infty} u_0(\cdot)$ ,  $\lim_{x \rightarrow -\infty} v_0(\cdot)$ , the positive lower bound of  $v_0(x)$  and the equation (0.7).

Proof: Now we consider  $(\tilde{u}, \tilde{v})(x, t, (m+1), k, u_\ell, v_\ell, u_r, v_r)$  defined in section 1 again and first remind some results in [2] and [7].

There is a  $c^2$  function  $f(x)$  satisfying

$$\begin{aligned} f(x) &= \text{constant} & \text{for } x \geq 0 \\ 0 < f'(x) < 1, f''(x) < 0, & \text{for } x < 0 \\ \lim_{x \rightarrow -\infty} f'(x) &= 1 \end{aligned} \quad (4.10)$$

such that  $(u_m, v_m)$  is connected to  $(u, v)$  by 1-wave if and only if

$$s_m - s_\ell = f(r_m - r_\ell) \quad (4.11)$$

where  $r_m - r_\ell \geq 0$  means that the 1-wave is a 1-rarefaction wave, and  $r_m - r_\ell < 0$  means that the 1-wave is a 1-shock wave; and  $(u_r, v_r)$  is connected to  $(u_\ell, v_\ell)$  by 2-wave on the left if and only if

$$v_r - r_m = f(s_r - s_m) \quad (4.12)$$

$s_r - s_m \geq 0$  ( $s_r - s_m < 0$ ) means that the 2-wave is a 2-rarefaction wave (a 2-shock wave respectively).



By the notation of (2.1) and (2.4) we have

$$\delta r = \Delta r + f(\Delta s) , \quad \delta s = \Delta s + f(\Delta r) \quad (4.13)$$

Because of (4.10) we get

$$\Delta r = p(\delta r, \delta s) , \quad \Delta s = q(\delta r, \delta s) \quad (4.14)$$

where  $p, q \in C^2$ . It is easy to know from simple computation that

$$\left\{ \begin{array}{ll} \frac{\partial p}{\partial(\delta r)} = \frac{1}{1-f'(p)f'(q)} & \frac{\partial q}{\partial(\delta r)} = \frac{-f'(p)}{1-f'(p)f'(q)} \\ \frac{\partial p}{\partial(\delta s)} = \frac{-f'(q)}{1-f'(p)f'(q)} & \frac{\partial q}{\partial(\delta s)} = \frac{1}{1-f'(p)f'(q)} \end{array} \right. \quad (4.15)$$

Suppose that  $(u_m, v_m)$  is connected to  $(u_\ell, v_\ell)$  by a 1-shock wave on the left. Since (0.5)(1.3) and (4.1), then we have

$$\left\{ \begin{array}{lll} s_m - s_\ell < (s_m + hS(s_m)) & -(s_\ell + hS(s_\ell)) & < 0 \\ r_m - r_\ell < (r_m + hR(r_m)) & -(r_\ell + hR(r_\ell)) & < 0 \end{array} \right. ,$$

$$\text{therefore } P((u_{\ell,h}, v_{\ell,h}), (u_{m,h}, v_{m,h})) < P(u_\ell, v_\ell), (u_m, v_m) \quad (4.16)$$

$$\text{where } u_{i,h} = u_i + U(u_i, v_i)h \quad v_{i,h} = v_i \exp\left(\frac{V(u_i, v_i)h}{v_i}\right) ,$$

$i$  can be  $\ell$ ,  $m$ , or  $r$ .

Suppose that  $(u_m, v_m)$  is connected to  $(u_l, v_l)$  by a 1-rarefaction wave on the left. Since (0.5)(1.3) and (4.1), then we have

$$s_m + hS(s_m) = s_l + S(s_l)$$

$$0 < r_m + hR(r_m) - (r_l + hR(r_l))$$

therefore

$$P((u_{l,h}, v_{l,h}), (u_{m,h}, v_{m,h})) = P((u_l, v_l), (u_m, v_m)) = 0 \quad (4.17)$$

The same kinds of results as (4.16) and (4.17) can be obtained for a 2-shock wave and a 2-rarefaction wave in  $(\tilde{u}, \tilde{v})$  respectively.

By combining the estimates (4.16), (4.17) and (2.3) we have

$$F((k+1)h-0) \leq F(kh+0) \quad \text{for } k \geq 0 \quad (4.18)$$

therefore for  $\lim_{V \rightarrow +\infty} r(\cdot, t)$  and  $\lim_{V \rightarrow +\infty} s(\cdot, t)$ , the decreasing variation of  $r(x, t)$  and  $s(x, t)$  on the interval  $(-\infty, +\infty)$ , the following inequalities hold

$$\begin{aligned} \lim_{V \rightarrow +\infty} r(\cdot, t) &\leq F(0) \leq \left( \lim_{V \rightarrow +\infty} r_0(\cdot) + \lim_{V \rightarrow +\infty} s_0(\cdot) \right) = Q_1 \\ \lim_{V \rightarrow +\infty} s(\cdot, t) &\leq F(0) \leq \left( \lim_{V \rightarrow +\infty} r_0(\cdot) + \lim_{V \rightarrow +\infty} s_0(\cdot) \right) = Q_1 \end{aligned} \quad (4.19)$$

It is not difficult to know from (4.20) that the strength of the shock wave in  $(\bar{u}, \bar{v})$  is less than  $Q_1$ , i.e.

$$|\Delta r| \leq Q_1 \quad \text{for} \quad 1\text{-shock wave} \quad (4.20)$$

$$|\Delta s| \leq Q_1 \quad \text{for} \quad 2\text{-shock wave} \quad (4.21)$$

Since (4.10) then

$$0 \leq c = f'(-Q_1) < 1 \quad (4.22)$$

Now we are ready to obtain the refinement of (4.16) suppose that  $(u_m, v_m)$  is connected to  $(u_\ell, v_\ell)$  by a 1-shock wave on the left. By (4.15) we have

$$\begin{aligned} P((u_{\ell,h}, v_{\ell,h}), (u_{m,h}, v_{m,h})) - P((u_\ell, v_\ell), (u_m, v_m)) = \\ \int_{\delta r}^{\delta r_h} \frac{1-f'(p)}{1-f'(p)f'(q)} d\xi + \int_{\delta s}^{\delta s_h} \frac{1-f'(q)}{1-f'(p)f'(q)} d\eta \geq (1-c)((\delta r_h - \delta r) + \\ + (\delta s_h - \delta s)) \geq (1-c)h [(R(r_m) - R(r_\ell) + (S(s_m) - S(s_\ell)))] \\ \geq - (1-c)\alpha h \Delta r \end{aligned} \quad (4.23)$$

$$\text{where } \delta r_k = (r_m + hR(r_m)) - (r_\ell + hR(r_\ell))$$

$$\delta s_h = (s_m + hS(s_m)) - (s_\ell + hS(s_\ell))$$

Suppose that  $(u_r, v_r)$  is connected to  $(u_m, v_m)$  by a 2-shock wave on the left. Similarly we have

$$\begin{aligned}
 P((u_{m,h}, v_{m,h}), (u_{r,h}, v_{r,h})) - P((u_m, v_m), (u_r, v_r)) &\geq \\
 &\geq -(1-c)\alpha h \Delta s \quad (4.24)
 \end{aligned}$$

By combining the estimates (4.23) (4.24) (4.17) and (2.3), we have

$$F(k+1)h-0 \leq (1-(1-c)\alpha h)F(kh+0) \quad . \quad \text{Q.E.D.}$$

We turn now to proving the existence of the weak solution of (0.1) and (0.2) and studying its asymptotic behavior. We use  $(u,v)(x,t)$  to denote the difference solutions of (0.1) and (0.2) before for convenience, although the difference solutions of (0.1) and (0.2) depend on the mesh length  $h$  and the sequence  $a$ . In order to distinguish between the difference solutions of (0.1) and (0.2) and the weak solution of (0.1) and (0.2), from now on we use  $(u^h, v^h)(x,t)$  to denote the difference solutions of (0.1) and (0.2) and  $(u,v)(x,t)$  to denote the weak solution of (0.1) and (0.2).

**Theorem 4.1.** Suppose that  $u_0(x), v_0(x)$  are bounded functions with bounded variation and do not depend  $x$  as  $|x| > M$ ,  $v_0(x)$  has a positive lower bound.  $U, V$  are smooth function satisfying (0.5), (0.6) (4.1) and (4.2) then the weak solution  $(u,v)(x,t)$  of (0.1) and (0.2) exists in  $t > 0$  and satisfies the following decay laws.

$$\lim_{t \rightarrow \infty} u(\cdot, t) \leq 2Q_1 e^{-(1-c)at} + Q_2 e^{-at} \quad (4.25)$$

$$\lim_{t \rightarrow \infty} \log v(\cdot, t) \leq 2Q_1 e^{-(1-c)at} + Q_2 e^{-at} \quad (4.26)$$

$$\left| u(x, t) - \frac{\bar{r} + \bar{s}}{2} \right| \leq 2Q_1 e^{-(1-c)at} + \frac{3}{2} Q_2 e^{-at} \quad (4.27)$$

$$\left| \log v(x, t) - \frac{\bar{r} - \bar{s}}{2} \right| \leq 2Q_1 e^{-(1-c)at} + \frac{3}{2} Q_2 e^{-at} \quad (4.28)$$

where  $Q_1 = \lim_{t \rightarrow \infty} r_0(\cdot) + \lim_{t \rightarrow \infty} s_0(\cdot)$ ,  $Q_2 = \frac{1}{2}(|r_0(-\infty) - \bar{r}| + |r_0(+\infty) - \bar{r}| + |s_0(-\infty) - \bar{s}| + |s_0(+\infty) - \bar{s}|)$ ,  $\bar{r}$  and  $\bar{s}$  are the unique solution of  $R(r) = 0$  and  $S(s) = 0$  respectively,  $c = f'(-Q_1)$ .

Proof. If we choose the ratio of the mesh lengths

$$\frac{l}{h} = \text{const} \geq e^{\frac{r^* + s^*}{2}} + 2Q_1$$

where  $r^* = \max(r_0(-\infty), \bar{r})$ ,  $s^* = \max(s_0(-\infty), \bar{s})$ , and  $h = 2^{-i}$ , where  $i$  is large enough such that  $h < \min(\frac{1}{\alpha}, \frac{1}{2K_1})$ , then the difference solution  $(u^h, v^h)(x, t)$  defined in section 1 can be defined in all half plan  $t > 0$  because that  $|r^h(-\infty, t)| \leq r^*$ ,  $|s^h(-\infty, t)| \leq s^*$  by Lemma 4.1 and  $\lim_{t \rightarrow \infty} r^h(\cdot, t) \leq 2Q_1$ ,  $\lim_{t \rightarrow \infty} s^h(\cdot, t) \leq 2Q_1$  by (4.19). By Lemma 3.1 and Lemma 3.2 we can obtain the existence of the weak solution  $(u, v)(x, t)$  of (0.1) and (0.2) in  $t > 0$  in similar way to that in theorem 3.1.

Using (4.9) and  $F(kh+0) \leq F(kh-0)$ ,  $k = 1, 2, \dots$ , we obtain

$$F(nh+0) \leq (1-(1-c)ah)^n F(0) = (1-(1-c)ah)^n Q_1,$$

then thus

$$\begin{cases} \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} r^h(\cdot, nh+0) \leq (1-(1-c)ah)^n Q_1 \\ \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} s^h(\cdot, nh+0) \leq (1-(1-c)ah)^n Q_1 \end{cases} \quad (4.29)$$

It is not difficult by the means similar to this in Lemma 4.2 to know that

$$\begin{cases} \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} r^h(\cdot, t) \leq \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} r^h(\cdot, nh+0) \\ \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} s^h(\cdot, t) \leq \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} s^h(\cdot, nh+0) \end{cases}, \quad \text{for } nh < t < (n+1)h. \quad (4.30)$$

let  $i$  tends to infinite, by (4.29) we have

$$\begin{cases} \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} r(\cdot, t+0) \leq Q_1 e^{-(1-c)at} \\ \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} s(\cdot, t+0) \leq Q_1 e^{-(1-c)at} \end{cases} \quad \begin{matrix} \text{for } t = j_2^{-k} \text{ and } k, j \text{ are any} \\ \text{positive integers} \end{matrix} \quad (4.31)$$

Therefore by (4.30) and (4.31) we obtain

$$\begin{cases} \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} r(\cdot, t) \leq Q_1 e^{-(1-c)at} \\ \begin{matrix} +\infty \\ DV \\ -\infty \end{matrix} s(\cdot, t) \leq Q_1 e^{-(1-c)at} \end{cases} \quad \begin{matrix} \text{for } t > 0 \end{matrix} \quad (4.32)$$

It follows from (4.6) that

$$|r^h(-\infty, t) - \bar{r}| \leq |r_0(-\infty) - \bar{r}| e^{-at}, \quad |r^h(+\infty, t) - \bar{r}| \leq |r_0(+\infty) - \bar{r}| e^{-at}$$

$$|s^h(-\infty, t) - \bar{s}| \leq |s_0(-\infty) - \bar{s}| e^{-at}, \quad |s^h(+\infty, t) - \bar{s}| \leq |s_0(+\infty) - \bar{s}| e^{-at}$$

therefore

$$|r(-\infty, t) - \bar{r}| \leq |r_0(-\infty) - \bar{r}| e^{-at}, \quad |r(+\infty, t) - \bar{r}| \leq |r_0(+\infty) - \bar{r}| e^{-at}$$

$$|s(-\infty, t) - \bar{s}| \leq |s_0(-\infty) - \bar{s}| e^{-at}, \quad |s(+\infty, t) - \bar{s}| \leq |s_0(+\infty) - \bar{s}| e^{-at}.$$

Thus then

$$\left\{ \begin{array}{l} \int_{-\infty}^{+\infty} r(\cdot, t) \leq 2Q_1 e^{-(1-c)at} + (|r_0(-\infty) - \bar{r}| + |r_0(+\infty) - \bar{r}|) e^{-at} \\ \int_{-\infty}^{+\infty} s(\cdot, t) \leq 2Q_1 e^{-(1-c)at} + (|s_0(-\infty) - \bar{s}| + |s_0(+\infty) - \bar{s}|) e^{-at} \\ |r(x, t) - \bar{r}| \leq \int_{-\infty}^{+\infty} r(\cdot, t) + |r_0(-\infty) - \bar{r}| e^{-at} \\ |s(x, t) - \bar{s}| \leq \int_{-\infty}^{+\infty} s(\cdot, t) + |s_0(-\infty) - \bar{s}| e^{-at} \end{array} \right. \quad (4.33)$$

The estimates (4.25), (4.26), (4.27) and (4.28) follow from (4.33).

Q.E.D.

## 5. Entropy Condition

The entropy condition for quasilinear hyperbolic systems of conservation laws has been extensively studied by Lax, P.D. [3], Dafermos, C.M [1] and Liu, Tai-Ping [4]. Here we will prove a theorem about entropy condition for the weak solution of (0.1) and (0.2) obtained in Section 3.

**Theorem 5.1.** Under the same assumptions of theorem 3.1, the weak solution  $(u,v)(x,t)$  of (0.1) and (0.2) constructed in theorem 3.1 satisfies the following entropy condition:

$$\iint_{0 < t < T} (g_t E + g_x F + g(uU - \frac{1}{v} V)) dx dt \geq 0 \quad (5.1)$$

for any non-negative smooth function  $g$  with compact support in  $0 < t < T$ ; where  $E(u,v) = \frac{u^2}{2} - \log v$ ,  $F(u,v) = \frac{u}{v}$ .

**Proof.** Suppose  $(u_m, v_m)$  is connected to  $(u_l, v_l)$  by a 1-shock wave with the speed of propagation  $s_1$  on the left in the solution of (0.7), let  $G(v_m, (u_l, v_l)) = s_1(E(u_m, v_m) - E(u_l, v_l)) - (F(u_m, v_m) - F(u_l, v_l))$ , then we have

$$G(v_l, (u_l, v_l)) = 0$$

$$\frac{\partial}{\partial v_m} G(v_m, (u_l, v_l)) > 0$$

because  $s_1 < 0$ ,  $v_m > v_l$  for a 1-shock wave in the weak solution of (0.7).



Therefore

$$G(v_m, (u_l, v_l)) > 0 \quad (5.2)$$

Suppose  $(u_r, v_r)$  is connected to  $(u_m, v_m)$  by a 2-shock wave with the speed of propagation  $s_2$  on the left in the solution of (0.7). Let  $G(v_r, (u_m, v_m)) = s_2(E u_r, v_r - E(u_m, v_m)) - (F(u_r, v_r) - F(u_m, v_m))$ , then similarly we have

$$G(v_r, (u_m, v_m)) > 0 \quad (5.2')$$

Using (3.3) (3.4) (3.9) and (3.10), we have

$$\iint_{t < 0 < T} (g_t E^h + g_x F^h + g(u^h u^h - \frac{v^h}{v^h}) dx dt = O(h) + I_1 + I_2 \quad (5.3)$$

Here

$$I_1 = \sum_{n=1}^{(\frac{T}{h})} g(x, nh) (E(u^h, v^h)(x, nh-0) - E(u^h, v^h)(x, nh+0)) dx \quad (5.4)$$

$$I_2 = \sum_{\substack{m+n=\text{even} \\ i=1,2}} \int_0^h g(s_{m,n,i} [E^h] - [F^h]) dt' \quad (5.5)$$

where  $s_{m,n,i}$  is the speed of propagation of the  $i$ -shock wave in  $(\tilde{u}, \tilde{v})(x, t(m+1), n, (u^h, v^h)(m+a_n)l, nh-0), (u^h, v^h)((m+2+a_n)l, nh-0)$ ,  $t' = t - nh$ .  $[ \cdot ]$  denotes the jump of the quantity in the bracket across the shock wave.  $E^h = E((u^h, v^h)(x, t))$ ,  $F^h = F((u^h, v^h)(x, t))$ .

Using (3.1) (3.2), we have

$$I_2 = O(h) + I'_2 \quad (5.6)$$

$$\text{where } I'_2 = \sum_{\substack{m+n=\text{even} \\ i=1,2}} \int_0^h g(s_{m,n,i} (E(\tilde{u}, \tilde{v})) - (F(\tilde{u}, \tilde{v}))) dt'$$

By (5.1), (5.2), we have

$$I'_2 \geq 0 \quad (5.7)$$

The entropy condition (5.1) follows from (5.3) (5.4) (5.5) (5.6) and (5.7).

Q.E.D.

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20. ABSTRACT, continued,

When the effects are strongly dissipative then the flow decays exponentially to a constant state as time tends to infinity.

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